Mössbauer effect for dark solitons in Bose-Einstein condensates

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(February 1, 2008)

We show that the energetic instability of dark solitons is associated with particle-like motion, and present a simple equation of motion, based on the Mössbauer effect, for dark solitons propagating in inhomogeneous Thomas-Fermi clouds. Numerical simulations support our theory. We discuss some experimental approaches.

PACS number(s): 03.75.Fi, 03.65 Ge

The success of the Gross-Pitaevski mean field theory in describing experimentally observed dilute Bose condensates [1–3] shows that one really can persuade a large number of particles to behave as a field. The perversely obvious thing to do with such a field, then, is to persuade it to behave as a particle. This would not only be a pleasant closing of a circle, but would link condensate physics to other subjects in which solitons are important, such as fluid mechanics, nonlinear optics, and fundamental particle theory. Perhaps most importantly, it might offer a powerful laboratory for studying the interface between quantum and classical physics, by providing a wholly artificial classical particle, composed of highly controllable, weakly interacting quantum particles. In this Letter we discuss one particular particle-like configuration of the Gross-Pitaevski mean field, namely the one-dimensional *dark soliton*. We describe the behaviour of dark solitons in inhomogeneous potentials, reporting both analytical and numerical results, and we propose some experimental approaches for effectively one-dimensional traps. We also suggest that the dark soliton should be considered the simplest member of a family of technically unstable but nevertheless robust monopoles in D dimensions.

The Gross-Pitaevski equation (GPE) governs the evolution of the c-number 'macroscopic wave function' $\psi(\vec{x},t)$ of a Bose-Einstein condensate. By appropriately scaling the wave function, space, and time, and incorporating a chemical potential by extracting a factor $e^{-i\kappa^2 t}$, one can write this equation in the convenient form

$$i\partial_t\psi = -\frac{1}{2}\nabla^2\psi + (|\psi|^2 + V(\vec{x}) - \kappa^2)\psi \; . \qquad (1)$$

We have assumed a positive scattering length, and have not restricted the normalization constant $\int dx\, |\psi|^2$. We will focus for most of this paper on the limit of a long trap so thin that one can apply the GPE in one dimension. The approach to this limit from three dimensions has recently been discussed [4].

Eqn. (1) in one dimension with $V = 0$ has been extensively studied in nonlinear optics [5], and a particle-like solution has long been known: $\psi_{DS} = \kappa\tanh(\kappa x)$. This time-independent solution is known as a *dark soliton*, because it describes a small dark spot in a light pulse. If $\psi(x)$ were restricted to be real, the dark soliton would be topologically stable in the same way as the Sine-Gordon kink that it strongly resembles. Since ψ is very nearly constant far from $x = 0$, but changes sign within a few healing lengths, it resembles a domain wall: a one-dimensional monopole.

There exist close relatives of the dark soliton which may be recognized as D-dimensional monopoles; we digress briefly to sketch the family group. If we allow ψ to be a multi-component object $\psi = (\psi_1, ..., \psi_D)$, and interpret $|\psi|^2 = \sum_j |\psi_j|^2$, then (1) is the mean field equation of motion for a condensate of atoms with D internal states (among which all scattering lengths are equal). In two dimensions one finds $\psi = f(\kappa r)(\cos\theta, \sin\theta)$ to be a solution to the two component GPE, where f is the radial profile of a one-component vortex. Analytical and numerical results related to this solution will be reported elsewhere. The generalization to three dimensions for three components is obvious [6]. In all these cases, if ψ were restricted to be real we would have topologically stable monopoles, as small 'bubbles' in the condensate.

It is not difficult to see, however, that by taking ψ into the complex plane one can continuously deform it into a configuration with constant density, eliminating the 'bubble' in $|\psi|^2$. One can exhibit this instability more precisely by exploring the free energy of nearby configurations; although this analysis can be carried out more generally, we now revert to one dimension. As a time-independent solution to (1) ($V = 0$, one dimension), $\psi_{DS} = \kappa\tanh\kappa x$ is a stationary point of the free energy functional

$$G = \frac{1}{2}\int dx\, \left[|\psi'|^2 + (|\psi|^2 - \kappa^2)^2\right] . \qquad (2)$$

One can diagonalize the Hessian matrix of G at the stationary point by solving what may be recognized as imaginary time Bogoliubov equations for the perturbation $\delta\psi$. (The real time Bogoliubov theory also diagonalizes the Hessian matrix, but by a symplectic instead of orthogonal transformation, as required by canonicity.) Because ψ is real, the real and imaginary parts of $\delta\psi = R + iS$ decouple, and independently satisfy (different) Schrödinger equations: the normal modes $R_k(x)$ and $S_k(x)$ are respectively eigenstates of $\hat{H}_3$ and $\hat{H}_1$, for

$$\hat{H}_n \equiv -\frac{1}{2}\frac{d^2}{dx^2} - \kappa^2 + n\psi_{DS}^2 \; . \qquad (3)$$

One finds the 'core bound state' $S_0 = 1/\cosh\kappa x$, for which the Hessian matrix has a negative eigenvalue; this

shows that there are configurations near the dark soliton with lower free energy. And in numerical evolution of the dark soliton in imaginary time, numerical noise soon finds this unstable mode, and fills in the monopole bubble.

Imaginary time evolution has only the most indirect implications for real time evolution, however. In one dimension with $V = 0$ constant, we have the following two parameter family of exact, but not generally time-independent, solutions to (1):

$$\psi_{q,\dot{q}}(x,t) = i\dot{q} + \sqrt{\kappa^2 - \dot{q}^2}\tanh\sqrt{\kappa^2 - \dot{q}^2}(x-q) \ , \quad (4)$$

where $\dot{q}$ is the time derivative of q, constant and satisfying satisfying $|\dot{q}| < \kappa$.

For these moving monopoles (familiar in nonlinear optics as 'grey solitons'), $|\psi|^2$ never drops below $\dot{q}^2$; and the phase slip across the soliton is $\pi - \arctan(|\dot{q}|/\sqrt{\kappa^2 - \dot{q}^2})$. This means that in the limit $\dot{q} \to \pm\kappa$, where the soliton is travelling at the superfluid critical velocity, the dark soliton actually vanishes, becoming identical with motionless condensate. But the free energy of the grey soliton is

$$G(q,\dot{q}) = \frac{4}{3}(\kappa^2 - \dot{q}^2)^{\frac{3}{2}} \ . \quad (5)$$

This implies that *grey solitons have negative kinetic energy.* The imaginary time Bogoliubov equations found only one negative energy degree of freedom near the dark soliton, and we can now identify it in the real time Bogoliubov theory as the canonical momentum conjugate to the translational zero mode. And so we see that *the instability of the dark soliton is not merely to break up or fill in, but to acquire velocity.*

Since dark solitons have both translational modes and independent velocities, they have the phase space of a particle. This is in contrast to a vortex in two dimensions, whose two positional degrees of freedom are in fact canonically conjugate, so that the vortex phase space is only two dimensional [7]. It is therefore natural to seek a particle-like second order equation of motion for dark solitons in an inhomogeneous potential $V(x)$. One can readily guess that, at least in a sufficiently slowly varying potential, one should simply replace $\kappa^2 \to \kappa^2 - V(q)$ in (4), and insert the result into a background Thomas-Fermi cloud. The question of how such a configuration will evolve in time, however, is not trivial. Time-dependent variational approaches can be informative but also misleading. Fortunately, a more systematic procedure exists.

In the experimentally relevant situation where $V(x)$ varies slowly on the healing length scale $|\psi|^{-1}$, one can apply a time-dependent boundary layer theory similar to that used for vortices in Ref. [7]. Perturb around the Thomas-Fermi approximation everywhere outside a small moving 'core zone' extending a few healing lengths from the monopole; treat the inhomogeneity of V as a perturbation within the core zone; and then match solutions smoothly at the moving boundary. This procedure leads to a very simple result in the case where the only excitations present (other than the soliton motion) are those generated by the soliton as it accelerates or decelerates. In this case the boundary layer theory demonstrates that when the healing length is much smaller than the Thomas-Fermi cloud, the system is in the Mössbauer limit, in which phonon mode excitations conserve momentum without contributing significantly to the total energy. As a result, the soliton energy $[\kappa^2 - \dot{q}^2 - V(q)]^{3/2}$ is conserved to leading order in the ratio of length scales, and this implies the equation of motion [8]

$$\ddot{q} = -\frac{1}{2}V'(q) \ . \quad (6)$$

In a harmonic trap, Eqn. (6) implies oscillation of the soliton [9] with frequency $1/\sqrt{2}$ times that of the dipole mode of the condensate. We have confirmed the frequency factor to rather more than the expected accuracy in numerical simulations of harmonic traps over a wide range of condensate densities and oscillation amplitudes; we have also monitored the center of mass motion, and confirmed that it is decoupled and has the trap frequency. Eqn. (6) also holds for arbitrary potentials, however, as long as they vary slowly on the healing length scale. We have therefore further confirmed the good accuracy of our Mössbauer limit equation of motion by solving Eqn. (1) numerically over a wide range of parameters and for various potentials; a generic sample is shown in Fig. 1.

To solve the NLSE numerically we use the split-step Fourier method [10], in two stages. First we construct an initial state with no excitations apart from the soliton, by propagating a crude trial wavefunction in imaginary time, and re-normalizing it between each step. Eventually, numerical noise excites the 'core bound state' and the monopole rapidly disappears in imaginary time; but before this there is a clear plateau in energy, during which the monopole slowly drifts in the Thomas-Fermi cloud, in the direction of decreasing density. We simply stop the imaginary time evolution during this period, and save the configuration. Then we propagate this optimized initial state in real time. As is well known, real time evolution is prone to (clearly recognizable) high frequency numerical instabilities; but by keeping our time steps shorter than h^2/π, where h is the spatial grid interval, we are able to simulate several periods of monopole motion, even at condensate densities approaching those attained in real experiments. (At $\int dx\, |\psi|^2 = 1000$, our Thomas-Fermi cloud extends to over ten times the trap width.)

Since Eqn. (6) predicts a particular frequency for finite amplitude oscillations around a local minimum of the trap potential, and a dynamical instability for motion near a local maximum, it would be interesting to compare these predictions with those from Bogoliubov theory. Since we have numerically found frequencies very close to that from Eqn. (6) even for extremely small amplitude oscillations, we strongly suspect that the two theories agree in the Mössbauer limit. Such agreement is not actually necessary, however: Eqn. (6) is derived for arbitrary amplitude oscillations but neglecting coupling to

phonon modes, while the Bogoliubov spectrum governs only infinitesimal perturbations, but includes all excitations. Co-incidence of the two spectra would imply that the dark soliton decouples from phonon modes at extrema of the potential, a result which might be of some importance for the problems of dark soliton zero point motion and mesoscopic tunneling.

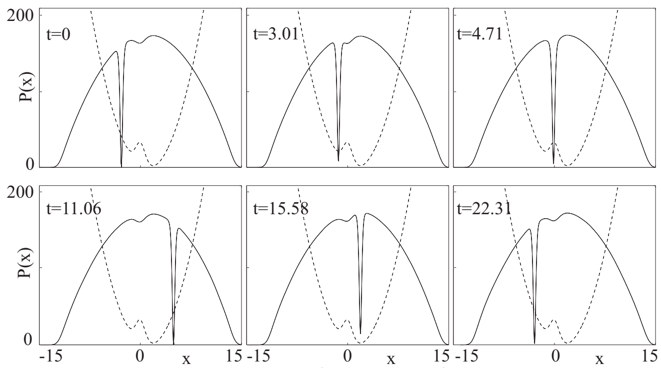

FIG. 1. Normalized $|\psi|^2$ ($\int dx\, |\psi|^2 = 300$) for a dark soliton oscillating in an asymmetric potential with a bump, shown (magnified) in dots. The potential is $V = 0.1x(x-2) + 1.1/\cosh^2(x)$, the soliton starting point is x=-2.81 (with a grid uncertainty of about 0.03); Eqn. (6) predicts the interval between successive times of complete darkness ($|\psi|^2 = 0$ at the minimum) to be $T/2 = 11.5 \pm 0.1$. Comparing times in lower left and lower right plots shows the Mössbauer limit is good to within 5%; the ratio of healing length $1/|\psi|$ to potential scale is of this order.

Since the equation of motion (6) is obtained by neglecting excitation of phonon modes by the monopole motion, but in fact some excitation does occur, we must expect dissipation (which could in principle be calculated by continuing the time-dependent boundary layer theory to higher orders in the ratio of length scales). Since the soliton energy is negative, however, dissipation has the effect of *increasing* the amplitude of the monopole motion in the trap. We have observed this effect in numerical simulations: phonon mode excitations are visible in slight distortions of the Thomas-Fermi envelope. These excitations produce small variations, plus a slow trend upwards, in $\dot{q}^2 + V(q)$; see Fig. 2.

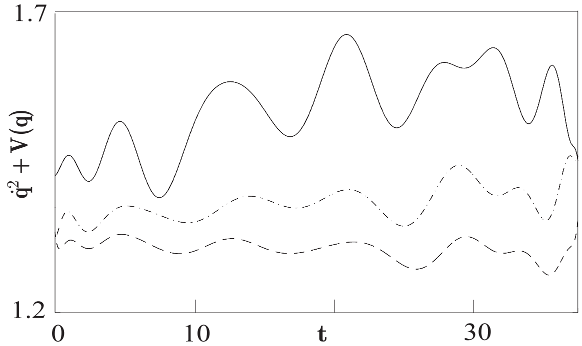

FIG. 2. Plots of the quantity $\dot{q}^2 + V(q)$, conserved in the Mössbauer limit, versus time, as a dark soliton moves in the potential of Fig. 1. The three curves are for various densities: $\int dx\, |\psi|^2 = 100, 500, 1000$ from top to bottom. For $\dot{q}^2$ we actually take the height of $|\psi|^2$ at the soliton minimum, which is equal to $\dot{q}^2$ for dark solitons in bulk.

We conclude our theoretical discusion by remarking on the interaction of two solitons. Analytical work on dark solitons in bulk has shown that two dark solitons interact through a short range repulsive potential, whose maximum height is finite and velocity dependent [11]. So solitons more than a few healing lengths apart do not influence one another, and solitons colliding rapidly can pass through each other with negligible interaction; but two slow solitons cannot pass each other. Moreover, the transition between impenetrability and near-freedom is fairly sharp as a function of relative velocity. On the other hand, the presence of a second soliton in a harmonic trap appears to *shorten* the oscillation period of the pair; see Fig. 3.

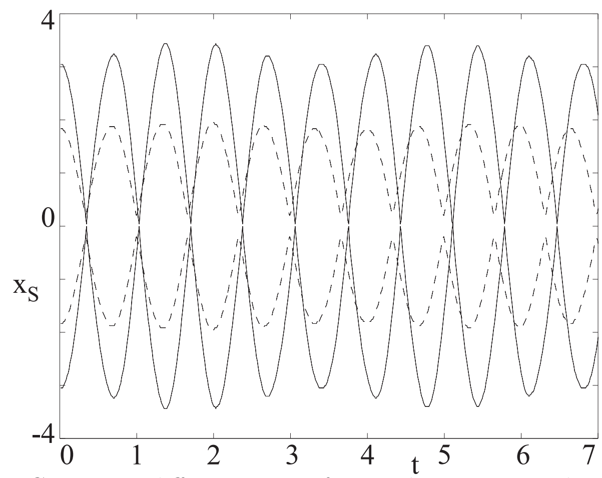

FIG. 3. Two different cases of two solitons oppositely displaced in a harmonic trap. The trap angular frequency is one, and the time axis is shown in units of 2π. The solid line shows solitons passing through each other; the dashed line, for two solitons initially displaced somewhat less, shows bouncing motion. For both cases, $\int dx\, |\psi|^2 = 100$. Note that the period of bouncing motion is approximately $2\pi \times 1.33$, that of passing motion $2\pi \times 1.36$. For comparison, the period of a single soliton at this density is $2\pi \times 1.42$. Both motions are also clearly modulated, due to coupling to a phonon mode.

We now briefly outline experimental approaches to the creation of dark solitons. An ingenious proposal has recently been presented to construct dark solitons by adiabatic passage [12]. This scheme requires that the constructed soliton be a dynamically stable stationary state, which would seem to prevent one from making dark solitons on slopes or local maxima of the potential. Since the capability exists to make temporary small local wells anywhere within the larger trap, however, this need not actually pose a problem. (This capability should also allow

more general testing of the dark soliton equation of motion, as one could arrange a sequence of ‘speed bumps’, or accelerating wells, along the thin trap.) Other recent work has noted that dark solitons are generically produced during collisions between condensates [13,14], or between generic travelling disturbances [15].

Another challenge is to detect dark solitons once they are made, since they are typically much smaller than a wavelength of light. Current techniques of relaxing the trap so that the soliton ‘hole’ grows to visibility should be helpful, as should the interferometric result that a dark soliton produces a discontinuity in the pattern of Ramsay fringes. We would like to suggest an alternative method, however, which could allow one to view dark solitons directly and non-destructively: trapping atoms with trapped atoms. Since there is a bound state in the dark soliton potential, a few atoms of another species could in principle be bound inside the dark soliton as it moves (assuming that they are repelled by the condensate atoms). In principle one might hope to introduce these probe atoms simply by ‘dusting’ the condensate with many of them, so that most probe atoms are simply repelled into the surrounding thermal cloud, but some are sympathetically cooled and ‘stick’ inside any dark solitons which may exist. Once there, they may be made to fluoresce independently of the condensate, so that the ‘dark’ solitons may be followed as small bright spots. Further investigation of this proposal is obviously required, as some condensate atoms are lost as the probe atoms are cooled sympathetically, and one must ensure that the capture cross section of the soliton is high enough that one does not need so much ‘dust’ that the soliton, or even the condensate, is destroyed. In principle however it seems a promising way to observe many types of topological defects.

As a final experimental remark, we note that in traps which are not sufficiently close to the one-dimensional limit one expects dark solitons to be unstable, presumably above some threshold velocity, to the formation of vortex rings; these would of course be at least as interesting as pure solitons.

In conclusion we wish to re-emphasize that although dark solitons are unstable, this instability should be much more of an advantage than a problem, as it allows the solitons to behave as particles, with independent position and velocity. In the Mössbauer limit a very simple equation of motion applies, which is expected to remain valid while the amplitude of soliton motion in the potential slowly grows. The final victory of the instability is the escape of the dark soliton from the trap: at least for these mesoscopic ‘hollow men’, the world will after all end with more of a bang than a whimper [16].

Acknowledgements

We are happy to acknowledge valuable discussions with Ignacio Cirac, Victor Perez-Garcia, Peter Zoller, and Wojciech Zurek. JRA is grateful for the hospitality of the Benasque Centre for Theoretical Physics, where part of this work was done. This work was supported by the European Union under the TMR Network ERBFMRX-CT96-0002 and by the Austrian Fond zur Förderung der wissenschaftlichen Forschung.

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